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Author(s)	KOJIMA, Sadayoshi
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FINITENESS OF SYMMETRIES ON 3-MANIFOLDS

Sadayoshi KOJIMA 小島定吉
(Tokyo Metropolitan University)

Since I have already reported this result in Kokyuroku # 487 of the Research Institute of Methematical Science and also already typed the manuscript [2], here I would like just to say how this work was motivated.

Let M be a smooth manifold. Then a finite subgroup of Diff M determines a finite group action on M. It is very natural to identify two actions if they arose from the conjugate subgroups in Diff M. There are quite many interesting open problems concerning finite group actions especially on 3-manifolds. As for the finiteness properties, Tollefson and Thurston asked fairly basic questions in Kirby's problem collection [1].

<u>Tollefson's question</u> : <u>Are there only finitely many conjugacy classes of finite cyclic subgroups of given order in</u> Diff M, <u>where</u> M <u>is a closed</u> 3-<u>manifold</u> ?

<u>Thurston's question</u> : <u>Is there an upper bound of the order of finite subgroups in</u> Diff M <u>when</u> M <u>does not admit a circle action</u> ?

Although both questions seem likely to be answered affirmatively, we have not had complete solutions yet. However we may try to apply the recent incredble developement of the 3-dimensional topology to some special cases of the questions. This is what I did and I actually used Thurston's theorem announced in [4]. Let us first review it.

There are precisely eight geometries needed for geometric structures on 3-manifolds. Those are the spherical geometry, euclidean geometry, hyperbolic geometry, nilpotent geometry, Lorentz geometry, solvable geometry and two product geometries. The well detailed description of these geometries can be found in Thurston's expository article [3], so I do not try to give their precise definition here.

In dimension 2, every closed manifolds indeed admits a geometric structure, but of course we can not expect such a fact for 3-manifolds. However surprisingly we can do expect the existence of a canonical decomposition of a 3-manifold each piece of which admits a geometric structure. Such a decomposition with geometric structures on each pieces is called a geometric decomposition and a 3-manifold which has a geometric decomposition is called geometric. Then Thurston has conjectured that every closed 3-manifold is geometric. This conjecture is true for a quite large class of 3-manifolds in the light of Thurston's work with Jaco-Shalen, Johannson's torus decomposition.

Let us now think of symmetries settled in the title. A symmetry on a manifold is a periodic automorphism with non empty fixed point set. Then Thurston's theorem announced in [4] is

Theorem (Thurston). *Let M be a closed orientable prime 3-manifold which admits an orientation preserving symmetry f. Then M admits a geometric decomposition with respect to which f is an isometry.*

In the original version, several conditions on the theorem are replaced by better ones, however we won't discuss them to avoid insignificant confusion. Also we won't try to figure out the proof.

Now Thurston's theorem reduces several topological questions concerning finite group actions to geometric ones. Actually if we restrict the questions by Tollefson and Thurston to actions generated by symmetries, all we need turns out to count up the number of all the isometric symmetries on geometric manifolds. This is exactly what I did and I got

Theorem A. *Let M be a closed orientable prime 3-manifold. Then there are only finitely many conjugacy classes of cyclic subgroups of given order in* Diff M *generated by an orientation preserving symmetry.*

Theorem B. Let M be a closed orientable irreducible non-spherical 3-manifold. Then the order of cyclic subgroups of Diff M generated by an orientation preserving symmetry is bounded.

Using Kneser-Milnor´s unique decomposition and Meeks-Simon-Yau´s equivariant sphere theorem, I can generalize these theorems for non prime closed manifolds without $S^1 \times S^2$ summand and for compact manifolds with toral boundaries.

Tollefson and Thurston´s questions were related with a property of 3-manifolds in general. However before such questions arose, there must be some supporting evidence as usual. For instanse, I could find not actually evidences but related conjectures in the knot theory which are very likely.

Montesinos Conjecture : No closed 3-manifold admits infinitely many involutions with orbit space S^3.

Fox-Sakuma Conjecture : Any non trivial knot (link) has only finitely many periods.

Here a symmetry of a link is by definition a symmetry of the link exterior which extends to a symmetry of S^3 and a period of a link is the order of an allowable symmetry of the link.

Then both conjectures are immediately follow from my theorems and they are now theorems. I must note that very recently Flapan and independently Hillman have proved the Fox-Sakuma conjecture without using Thurston's theorem but using the least area surface theory of Freedman-Hass-Scott.

References

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[4] Thurston, W. : Three manifolds with symmetry, preliminary report (1982).